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A higher order approximate static condensation method for multi-material diffusion problems

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SIAM CSE, 27 Feb 2019

Overview

1. The ASC(n) Method
   - Problem Setting
   - Description of the Method
   - ASC(0) and ASC(1)
2. Numerical Experiments
   - ASC(0) $\rightarrow$ ASC(1): Motivation
   - Piecewise P_1 & P_2 Solutions
   - Unsteady Problem

Diffusion Problem

Our objective is to solve the diffusion problem in the mixed form

$$\begin{cases} \mathbf{K}^{-1}\,\mathbf{u} + \nabla\, p = 0 & \text{in } \Omega \subset \mathbb{R}^2, \\ \nabla \cdot \mathbf{u} + c\, p \;= f & \text{in } \Omega, \end{cases}$$

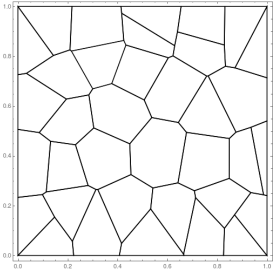
with boundary data

$$\begin{aligned} p &= g_D \quad \text{on } \partial\Omega_D, \\ \mathbf{u} \cdot \hat{\mathbf{n}} &= g_N \quad \text{on } \partial\Omega_N. \end{aligned}$$

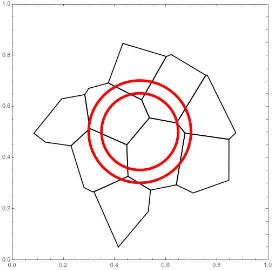
Challenges:

- The diffusion tensor **K** may sharply vary in Ω and may be discontinuous
- We want to use general polygonal meshes, and
- be able to handle material interfaces not aligned with the mesh

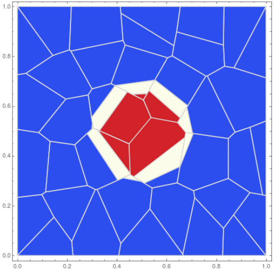
Mesh



(a) Macro-mesh $\mathscr{T}$



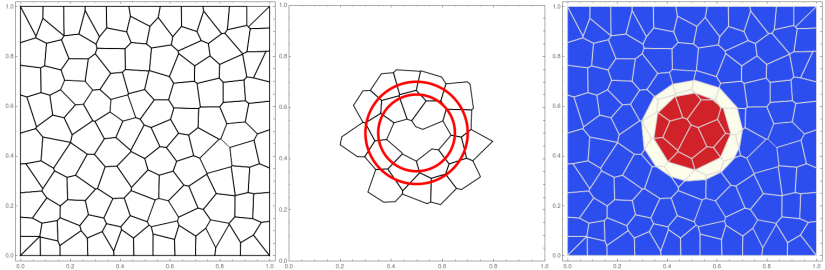
(b) Multi-material cells



(c) MOF

Moment-of-fluid interface reconstruction $\Rightarrow$ reconstructed interface may be discontinuous

Mesh



(a) Macro-mesh $\mathcal{T}$ (b) Multi-material cells (c) MOF

Moment-of-fluid interface reconstruction $\Rightarrow$ reconstructed interface may be discontinuous

Local Problem

Consider $T \in \mathcal{T}_H$: $\begin{cases} \mathbf{K}^{-1}\,\mathbf{u} + \nabla\, p = 0 & \text{in } T, \\ \nabla \cdot \mathbf{u} + c\, p \;= f & \text{in } T, \\ p \;= \lambda & \text{on } \partial T \end{cases}$

$$\Downarrow$$

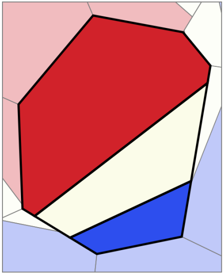
Find trial functions $\langle \mathbf{u}, p \rangle \in \mathbb{H}_{\text{div}}(T) \times \mathbb{L}^2(T)$ such that

$$\begin{cases} \displaystyle\int_T \mathbf{K}^{-1}\,\mathbf{u} \cdot \mathbf{v}\,\mathrm{d}\mathbf{x} - \int_T p\,\nabla \cdot \mathbf{v}\,\mathrm{d}\mathbf{x} = -\int_{\partial T} \lambda\,\mathbf{v} \cdot \hat{\mathbf{n}}\,\mathrm{d}l, \\ \displaystyle\int_T \nabla \cdot \mathbf{u}\, q\,\mathrm{d}\mathbf{x} + \int_T c\, p\, q\,\mathrm{d}\mathbf{x} = \int_T f\, q\,\mathrm{d}\mathbf{x} \end{cases}$$

holds for all test functions $\langle \mathbf{v}, q \rangle \in \mathbb{H}_{\text{div}}(T) \times \mathbb{L}^2(T)$

Consider $T \in \mathcal{T}_H$:
$$\begin{cases} \mathbf{K}^{-1}\mathbf{u} + \nabla p = 0 & \text{in } T, \\ \nabla \cdot \mathbf{u} + c\,p = f & \text{in } T, \\ p = \lambda & \text{on } \partial T \end{cases}$$

$$\Downarrow$$



Minimesh $\boldsymbol{\tau}_h$ of T

$\Rightarrow$

Discretization

Apply Mimetic Finite Difference Method*

$$\Downarrow$$

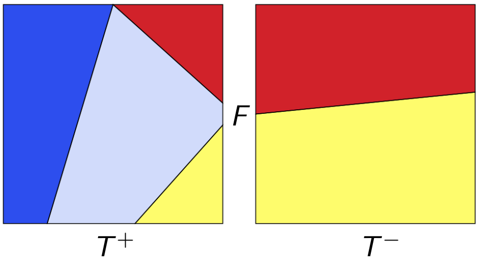
$$\begin{pmatrix} \mathbf{M}_\tau & \mathbf{B}_\tau^T \\ -\mathbf{B}_\tau & \boldsymbol{\Sigma}_\tau \end{pmatrix} \begin{pmatrix} \mathbf{u}_\tau \\ \mathbf{p}_\tau \end{pmatrix} = \begin{pmatrix} \mathbf{E}_\tau\, \mathbf{C}_\tau\, \boldsymbol{\lambda}_\tau \\ \mathbf{f}_\tau \end{pmatrix}$$

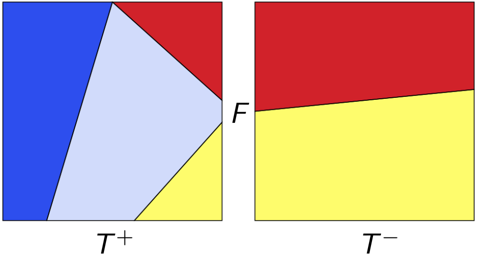
*L. Beirao da Veiga, K. Lipnikov, G. Manzini
The Mimetic Finite Difference Method for Elliptic Problems
Springer 2014

If one knows the pressure trace λ for each $T \in \mathcal{T}$, one can recover the solution in $\mathcal{T}$. The idea is **(i)** to express external flux DOFs in terms of trace DOFs (*static condensation*),

$$\mathbf{u}_{\boldsymbol{\tau}}^{\text{ext}} := \mathbf{E}_{\boldsymbol{\tau}}^{T}\,\mathbf{u}_{\boldsymbol{\tau}} = \mathbf{A}_{\boldsymbol{\tau}}\,\mathbf{C}_{\boldsymbol{\tau}}\,\boldsymbol{\lambda}_{\boldsymbol{\tau}} - \mathbf{a}_{\boldsymbol{\tau}},$$

and **(ii)** to get the system for trace DOFs by requiring weak continuity of fluxes. **Problem**: we may have different number of trace DOFs from T^+ and T^-





Solution: approximate a pressure trace on F with a polynomial $\lambda_F \in \mathbb{P}^n(F)$ described in terms of its $(n+1)$ moments

$$\lambda_F^{(i)} = \frac{\int_F \lambda\, s_i \,\mathrm{d}l}{|F|}, \quad i = 0, \dots, n.$$

Here $s_i \in \mathbb{P}^i(F)$ is a fixed polynomial of degree i such that $s_i \perp_{\mathbb{L}^2} s_j$, $j < i$

ASC(n): DOFs and Constraints

DOFs

Now we express trace DOFs on mini faces of $\boldsymbol{\tau}$ via coarse trace DOFs := $(n+1)$ moments on each base face of T,

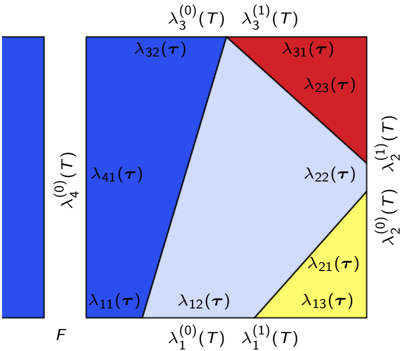
$$\begin{aligned} \boldsymbol{\lambda}_{\boldsymbol{\tau}} &= \mathbf{R}_{\boldsymbol{\tau}}\, \boldsymbol{\lambda}_T \quad \Rightarrow \\ \mathbf{u}_{\boldsymbol{\tau}}^{\text{ext}} &= \mathbf{A}_{\boldsymbol{\tau}}\, \mathbf{C}_{\boldsymbol{\tau}}\, \mathbf{R}_{\boldsymbol{\tau}}\, \boldsymbol{\lambda}_T - \mathbf{a}_{\boldsymbol{\tau}}, \end{aligned}$$

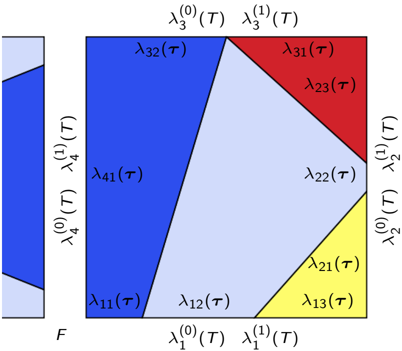
Constraints

and close the system by requiring weak continuity of normal fluxes on each base face

$$\int_F \mathbf{u}|_{T^+} \cdot \hat{\mathbf{n}}\, s_i \,\mathrm{d}l = \int_F \mathbf{u}|_{T^-} \cdot \hat{\mathbf{n}}\, s_i \,\mathrm{d}l, \; i = 0, \ldots, n \text{ for } F \in \mathcal{F}_{\text{int}}$$

Express fluxes in terms of traces $\Rightarrow$ get SLAE for coarse trace DOFs





$$
\underbrace{\begin{pmatrix} \lambda_{11}(\boldsymbol{\tau}) \\ \lambda_{12}(\boldsymbol{\tau}) \\ \lambda_{13}(\boldsymbol{\tau}) \\ \lambda_{21}(\boldsymbol{\tau}) \\ \lambda_{22}(\boldsymbol{\tau}) \\ \lambda_{23}(\boldsymbol{\tau}) \\ \lambda_{31}(\boldsymbol{\tau}) \\ \lambda_{32}(\boldsymbol{\tau}) \\ \lambda_{41}(\boldsymbol{\tau}) \end{pmatrix}}_{\boldsymbol{\lambda}_{\boldsymbol{\tau}}=}
=
\underbrace{\begin{pmatrix}
1 & s_{11} & 0 & 0 & 0 & 0 & 0 \\
1 & s_{12} & 0 & 0 & 0 & 0 & 0 \\
1 & s_{13} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & s_{21} & 0 & 0 & 0 \\
0 & 0 & 1 & s_{22} & 0 & 0 & 0 \\
0 & 0 & 1 & s_{23} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & s_{31} & 0 \\
0 & 0 & 0 & 0 & 1 & s_{32} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}}_{\mathbf{R}_{\boldsymbol{\tau}}=}
\underbrace{\begin{pmatrix} \lambda_1^{(0)}(T) \\ \lambda_1^{(1)}(T) \\ \lambda_2^{(0)}(T) \\ \lambda_2^{(1)}(T) \\ \lambda_3^{(0)}(T) \\ \lambda_3^{(1)}(T) \\ \lambda_4^{(0)}(T) \end{pmatrix}}_{\boldsymbol{\lambda}_T=}
$$

Here $s_{ij} :=$ signed distance between jth mini-face and ith macro-face centroids. For ASC(0) the matrix $\mathbf{R}_{\boldsymbol{\tau}}$ is as on the left with rows containing s_{ij}-s eliminated.

$$\int_F \mathbf{u}|_{T^+} \cdot \hat{\mathbf{n}}\, s_i \,\mathrm{d}l = \int_F \mathbf{u}|_{T^-} \cdot \hat{\mathbf{n}}\, s_i \,\mathrm{d}l, \; i = 0, \dots, n \text{ for } F \in \mathcal{F}_{\text{int}}$$

$$\Downarrow$$

$$n = 0: \quad \sum_{f \in f_F(T^+)} u^{\text{ext}}_{\tau^+}(f)\,|f| + \sum_{f \in f_F(T^-)} u^{\text{ext}}_{\tau^-}(f)\,|f| = 0,$$

$$n = 1: \quad \sum_{f \in f_F(T^+)} u^{\text{ext}}_{\tau^+}(f) \int_f s_1 \,\mathrm{d}l + \sum_{f \in f_F(T^-)} u^{\text{ext}}_{\tau^-}(f) \int_f s_1 \,\mathrm{d}l = 0$$

$$\Downarrow$$

$$\left(\mathbf{R}^T_{\tau^+}\,\mathbf{C}_{\tau^+}\,\mathbf{u}^{\text{ext}}_{\tau^+}\right)_{i+m} + \left(\mathbf{R}^T_{\tau^-}\,\mathbf{C}_{\tau^-}\,\mathbf{u}^{\text{ext}}_{\tau^-}\right)_{j+m} = 0, \quad m \in \{0, 1\}$$

$$\Downarrow$$

$$\Bigg(\underbrace{\left(\mathbf{R}^T_{\tau^+}\,\mathbf{C}_{\tau^+}\,\mathbf{A}_{\tau^+}\,\mathbf{C}_{\tau^+}\,\mathbf{R}_{\tau^+}\right)}_{\mathbf{S}_{T^+} :=} \boldsymbol{\lambda}_{T^+}\Bigg)_{i+m} + \Bigg(\underbrace{\left(\mathbf{R}^T_{\tau^-}\,\mathbf{C}_{\tau^-}\,\mathbf{A}_{\tau^-}\,\mathbf{C}_{\tau^-}\,\mathbf{R}_{\tau^-}\right)}_{\mathbf{S}_{T^-} :=} \boldsymbol{\lambda}_{T^-}\Bigg)_{j+m} =$$

$$\underbrace{\left(\mathbf{R}^T_{\tau^+}\,\mathbf{C}_{\tau^+}\,\mathbf{a}_{\tau^+}\right)}_{\mathbf{s}_{T^+}}{}_{i+m} + \underbrace{\left(\mathbf{R}^T_{\tau^-}\,\mathbf{C}_{\tau^-}\,\mathbf{a}_{\tau^-}\right)}_{\mathbf{s}_{T^-}}{}_{j+m}$$

ASC(1) System Assembly

$$\int_F \mathbf{u}|_{T^+} \cdot \hat{\mathbf{n}}\, s_i \,\mathrm{d}l = \int_F \mathbf{u}|_{T^-} \cdot \hat{\mathbf{n}}\, s_i \,\mathrm{d}l,\; i = 0, \ldots, n \text{ for } F \in \mathscr{F}_{\text{int}}$$

$$\Downarrow$$

$$n = 0: \quad \sum_{f \in f_F(T^+)} u^{\text{ext}}_{\boldsymbol{\tau}^+}(f)\,|f| + \sum_{f \in f_F(T^-)} u^{\text{ext}}_{\boldsymbol{\tau}^-}(f)\,|f| = 0,$$

$$n = 1: \quad \sum_{f \in f_F(T^+)} u^{\text{ext}}_{\boldsymbol{\tau}^+}(f) \int_f s_1 \,\mathrm{d}l + \sum_{f \in f_F(T^-)} u^{\text{ext}}_{\boldsymbol{\tau}^-}(f) \int_f s_1 \,\mathrm{d}l = 0$$

$$\Downarrow$$

$$\left(\mathbf{R}^T_{\boldsymbol{\tau}^+}\, \mathbf{C}_{\boldsymbol{\tau}^+}\, \mathbf{u}^{\text{ext}}_{\boldsymbol{\tau}^+}\right)_{i+m} + \left(\mathbf{R}^T_{\boldsymbol{\tau}^-}\, \mathbf{C}_{\boldsymbol{\tau}^-}\, \mathbf{u}^{\text{ext}}_{\boldsymbol{\tau}^-}\right)_{j+m} = 0, \quad m \in \{0, 1\}$$

$$\Downarrow$$

$$\mathbf{S}_{\mathcal{T}} = \sum_{T \in \mathcal{T}} \mathbf{N}_T^T\, \mathbf{S}_T\, \mathbf{N}_T,$$

$$\mathbf{s}_{\mathcal{T}} = \sum_{T \in \mathcal{T}} \mathbf{N}_T^T\, \mathbf{s}_T,$$

Global system:

$$\mathbf{S}_{\mathcal{T}}\, \boldsymbol{\lambda}_{\mathcal{T}} = \mathbf{s}_{\mathcal{T}}$$

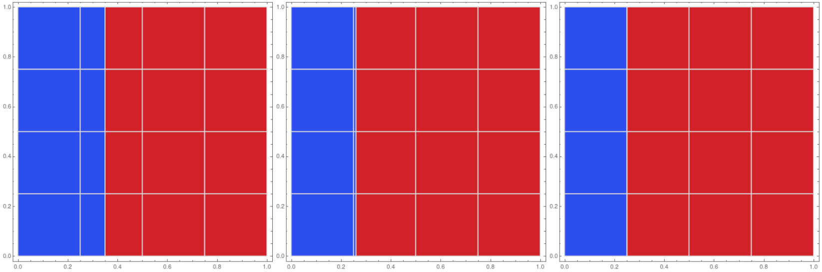
ASC(1) System Assembly

$$\mathbf{S}_{\mathcal{T}} = \sum_{T\in\mathcal{T}} \mathbf{N}_T^T\,\mathbf{S}_T\,\mathbf{N}_T, \qquad \text{Global system:}$$

$$\mathbf{s}_{\mathcal{T}} = \sum_{T\in\mathcal{T}} \mathbf{N}_T^T\,\mathbf{s}_T, \qquad \mathbf{S}_{\mathcal{T}}\,\boldsymbol{\lambda}_{\mathcal{T}} = \mathbf{s}_{\mathcal{T}}$$

- **Theorem**: system matrix $\mathbf{S}_{\mathcal{T}}$ is sparse and SPD for ASC(0) and ASC(1)
- Hence efficient solvers and preconditioners are available (e. g. CG + Algebraic Multigrid)
- Once we obtain $\boldsymbol{\lambda}_{\mathcal{T}}$, we recover pressure and flux DOFs in each cell $T \in \mathcal{T}$ (this may be done in parallel)

Figure: $w :=$ width of the left minimesh cells

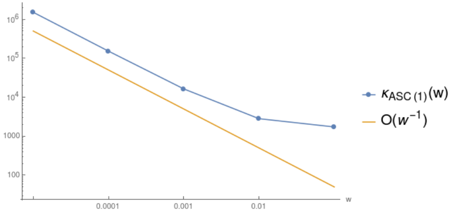


(a) $w = .1$ (b) $w = .01$ (c) $w = .001$

We solve the diffusion problem w/ $\mathbf{K} = k\,\mathbf{I}$, $k = 1$ on the left part and .1 on the right. Exact solution is piecewise linear

Figure: Condition Numbers of ASC(0) / ASC(1) Matrices

w	$\kappa_{\text{ASC}(0)}$	$\kappa_{\text{ASC}(1)}$	$\widetilde{\kappa}_{\text{ASC}(1)}$
10^{-1}	41.0	1 730	41.0
10^{-2}	45.2	2 817	45.1
10^{-3}	48.3	16 391	48.3
10^{-4}	49.0	152 325	49.0
10^{-5}	49.1	1.5×10^6	49.1



$\kappa_{\text{ASC}(0)}$ does not depend on w, and $\kappa_{\text{ASC}(1)}$ is proportional to w^{-1}. However, if we remove 3 smallest eig values (corresponding to 3 int MM faces), **we will have $\tilde{\kappa}_{\text{ASC}(1)} \approx \kappa_{\text{ASC}(0)}$**. Starting from some iteration CG behaves like extreme eig values are not present; that is, several small eig values is not a problem

ℓ^2-Error

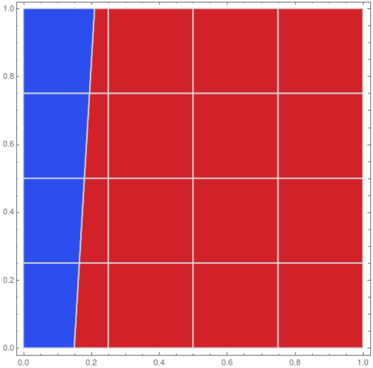
If the base mesh consists of triangles + we have no material interfaces, ASC(n) boils down to Mixed-Hybrid Raviart–Thomas FEM:

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbb{L}^2(\Omega)} \leq c\, h\, \|\mathbf{u}\|_{\mathbb{H}^1(\Omega)},$$
$$\|p - p_h\|_{\mathbb{L}^2(\Omega)} \leq c \left(h\, \|p\|_{\mathbb{H}^1(\Omega)} + h^2\, \|p\|_{\mathbb{H}^2(\Omega)} \right).$$

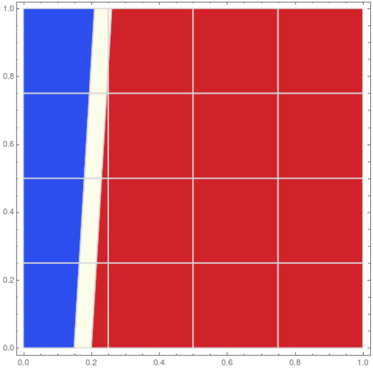
That is, we cannot expect ASC(n) convergence to be better than linear. We define **discrete $\mathbb{L}^2$-norm**

$$\|v\|_{\ell^2(\Omega)} := \|P_h\, v\|_{\mathbb{L}^2(\Omega)} \leq \|v\|_{\mathbb{L}^2(\Omega)},$$

where $P_h :=$ $\mathbb{L}^2$-projection operator on the space of piecewise constant functions on each cell $T \in \mathcal{T}$ (or on each $\tau \in \boldsymbol{\tau}$ if T is a MMC)



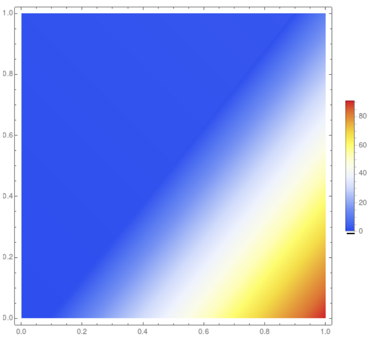
(a) $||p - p_h||_{\ell^2(\Omega)} = 6.38 \times 10^{-2}$



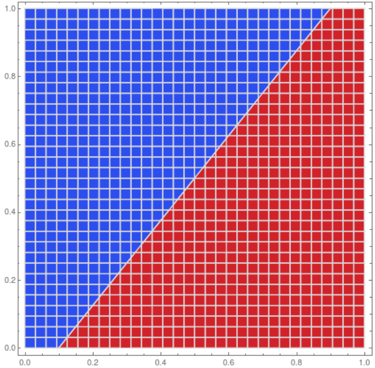
(b) $||p - p_h||_{\ell^2(\Omega)} = 6.41 \times 10^{-2}$

Here $\mathbf{K}_i = \mathbf{K}_j$ and the exact soln is linear. ASC(0) produces errors due to const trace approximation, and ACS(1) recovers the exact soln

We solve the diffusion problem on the sequence of square meshes w/ $\mathbf{K} = k\,\mathbf{I}$, $k = 1$ on the left part and .1 on the right. Exact solution is piecewise linear



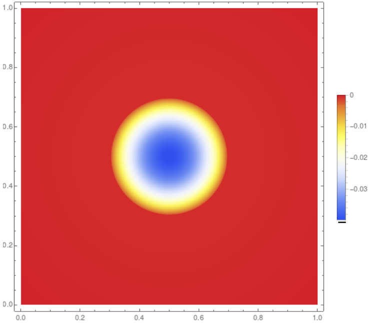
(a) Exact soln, p



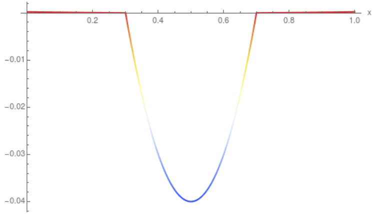
(b) Materials

	h	$e_p^{\ell^2}$	ρ_p	e_p^{∞}	$e_u^{\ell^2}$	ρ_u
ASC(0)	3.5×10^{-1}	7.3×10^{-1}		4.8	6.6×10^{-1}	
	8.8×10^{-2}	1.6×10^{-1}	1.1	1.2	3.5×10^{-1}	0.46
	2.2×10^{-2}	3.7×10^{-2}	1.1	3.4×10^{-1}	1.3×10^{-1}	0.71
	5.5×10^{-3}	8.9×10^{-3}	1.0	7.9×10^{-2}	4.1×10^{-2}	0.83
	h	$e_p^{\ell^2}$	ρ_p	e_p^{∞}	$e_u^{\ell^2}$	ρ_u
ASC(1)	3.5×10^{-1}	2.5×10^{-2}		2.9×10^{-1}	4.6×10^{-2}	
	8.8×10^{-2}	1.9×10^{-3}	1.84	6.6×10^{-2}	2.0×10^{-2}	0.6
	2.2×10^{-2}	1.6×10^{-4}	1.79	4.3×10^{-2}	5.5×10^{-3}	0.93
	5.5×10^{-3}	1.3×10^{-5}	1.80	2.0×10^{-2}	1.3×10^{-3}	1.

We solve the diffusion problem on Voronoi meshes w/ $\mathbf{K} = k\,\mathbf{I}$, $k = 1$ outside the circle and .001 inside. Exact solution is pw quadratic. We compare convergence of ASC(0) and ASC(1)



(a) Exact soln, p



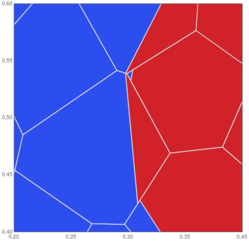
(b) $p(x, \frac{1}{2})$

Piecewise Quadratic Solution w/ 2 Materials (2 / 3)

	h	$e_p^{\ell^2}$	ρ_p	e_p^{∞}	$e_u^{\ell^2}$	ρ_u
ASC(0)	3.0×10^{-1}	2.4×10^{-3}		6.3×10^{-1}	6.9×10^{-5}	
	1.5×10^{-1}	6.5×10^{-4}	2.0	7.0×10^{-3}	3.6×10^{-5}	0.9
	8.1×10^{-2}	2.6×10^{-4}	1.4	3.2×10^{-3}	3.4×10^{-5}	0.09
	4.2×10^{-2}	1.4×10^{-4}	0.9	2.3×10^{-3}	2.1×10^{-5}	0.73
	2.1×10^{-2}	3.7×10^{-5}	1.9	1.1×10^{-3}	1.1×10^{-5}	0.93
	1.0×10^{-2}	2.7×10^{-5}	0.4	8.6×10^{-4}	6.5×10^{-6}	0.75

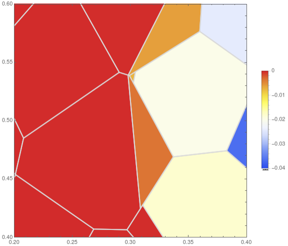
	h	$e_p^{\ell^2}$	ρ_p	e_p^{∞}	$e_u^{\ell^2}$	ρ_u
ASC(1)	3.0×10^{-1}	2.4×10^{-3}		2.1×10^{-3}	1.2×10^{-4}	
	1.5×10^{-1}	7.0×10^{-4}	1.9	1.3×10^{-2}	6.5×10^{-5}	0.88
	8.1×10^{-2}	2.3×10^{-4}	1.8	6.8×10^{-4}	3.0×10^{-5}	1.25
	4.2×10^{-2}	6.8×10^{-5}	1.8	3.2×10^{-4}	1.4×10^{-5}	1.16
	2.1×10^{-2}	2.0×10^{-5}	1.8	1.1×10^{-4}	5.0×10^{-6}	1.48
	1.0×10^{-2}	5.4×10^{-6}	1.9	3.3×10^{-5}	2.2×10^{-6}	1.18

We observe a jump of ∞-error of ASC(1) at $h = 1.5 \times 10^{-1}$

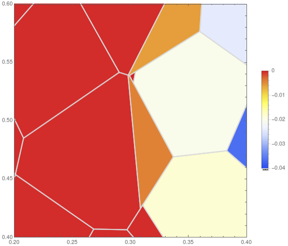


(a) Materials

$h = 1.5 \times 10^{-1}$: This example shows that ASC(1) ∞-norm may be sensitive to geometry errors. However, it does not affect ℓ^2-convergence

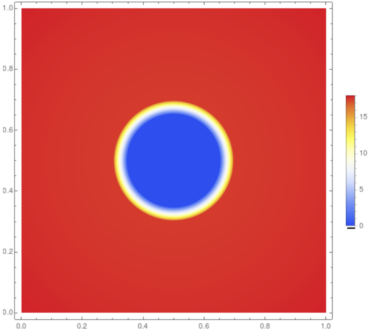


(b) ASC(0), p_h

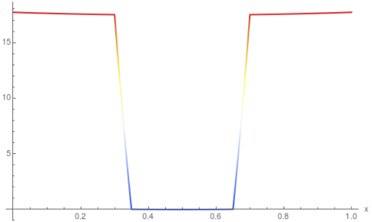


(c) ASC(1), p_h

We solve the diffusion problem on triangular meshes w/ $\mathbf{K} = k\,\mathbf{I}$, $k = 1$ outside the ring and .001 inside. Exact solution is piecewise quadratic



(a) Exact soln, p



(b) $p(x, \frac{1}{2})$

Piecewise Quadratic Solution w/ 3 Materials (2 / 2)

ASC(0)	h	$e_p^{\ell^2}$	ρ_p	e_p^∞
	3.0×10^{-1}	4.5		17
	2.5×10^{-1}	4.5		17
	1.3×10^{-1}	4.0		17
	8.3×10^{-2}	4.4		17
	6.7×10^{-2}	7.1×10^{-1}		4.9
	4.3×10^{-2}	4.5×10^{-1}	1.2	5.0

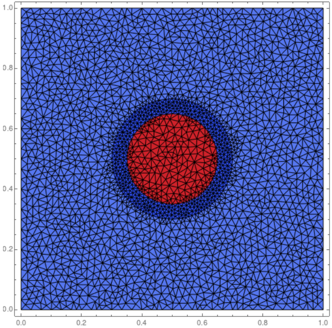
Arithmetic homo.	h	$e_p^{\ell^2}$	ρ_p	e_p^∞
	3.0×10^{-1}	4.9		17
	2.5×10^{-1}	5.0		17
	1.3×10^{-1}	4.9		17
	8.3×10^{-2}	4.7		17
	6.7×10^{-2}	4.4		16
	4.3×10^{-2}	9.7×10^{-1}	3.5	5.7

ASC(1)	h	$e_p^{\ell^2}$	ρ_p	e_p^∞
	3.0×10^{-1}	4.5×10^{-1}		3.5
	2.5×10^{-1}	2.6×10^{-1}	3	2.7
	1.3×10^{-1}	9.2×10^{-2}	1.5	6.2×10^{-1}
	8.3×10^{-2}	4.8×10^{-2}	1.6	8.3×10^{-1}
	6.7×10^{-2}	2.8×10^{-2}	2.5	2.3×10^{-1}
	4.3×10^{-2}	1.0×10^{-2}	2.3	6.3×10^{-2}

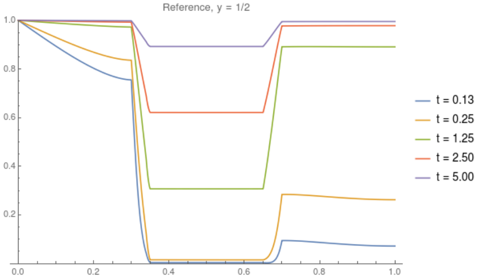
Harmonic homo.	h	$e_p^{\ell^2}$	ρ_p	e_p^∞
	3.0×10^{-1}	2.3		15
	2.5×10^{-1}	1.7	1.6	16
	1.3×10^{-1}	7.3×10^{-1}	1.2	12
	8.3×10^{-2}	4.8×10^{-1}	1.0	12
	6.7×10^{-2}	3.4×10^{-1}	1.6	9.4
	4.3×10^{-2}	1.6×10^{-1}	1.7	8.2

Before $h = 6.7 \times 10^{-2}$ we have cells / faces with 3 materials, and after this mesh level we have only 2 material MMCs

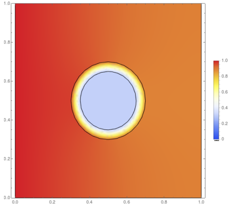
We solve the unsteady diffusion problem on Voronoi meshes w/ $\mathbf{K} = k\,\mathbf{I}$, $k = .002$ inside the ring, 10 in the inner disk, and 1 elsewhere. We set $g_D = 1$ on the left bndry and 0 on the right; top and bottom bndries are insulated. Equilibrium state is $p \equiv 1$



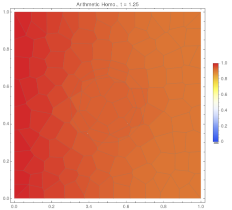
(a) Conforming (super)mesh



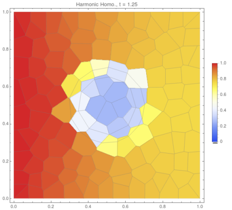
(b) Cuts $p_*\big((x, 0.5), t\big)$ of the ref soln



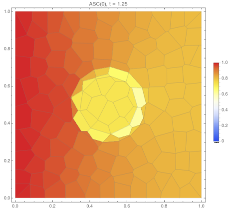
(a) Reference



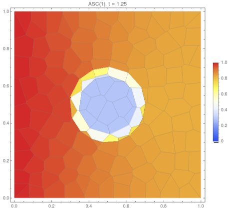
(b) Arithmetic homo.



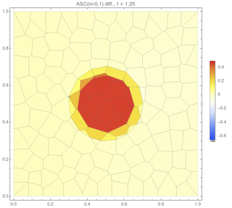
(c) Harmonic homo.



(d) ASC(0)



(e) ASC(1)



(f) ASC(0, 1) difference

Comparison of the discrete solutions p_h, $h = 1.5 \times 10^{-1}$, $t = 1.25$ (play)

Results:

- ASC(n) is able to efficiently handle unfitted material interfaces
- 2^{nd} order ℓ^2-convergence for ASC(1)
- Effective condition number seems to be uniformly bounded w.r.t. an interface position
- The underline matrix is SPD and sparse; its pattern does not depend on mini meshes
- A. Zhiliakov et al. A higher order approximate static condensation method for multi-material diffusion problems, 2019 (JCP preprint)

TODO List:

- Anisotropic diffusion: homogenization is not applicable; what about ASC(n)?